Fourier transformation

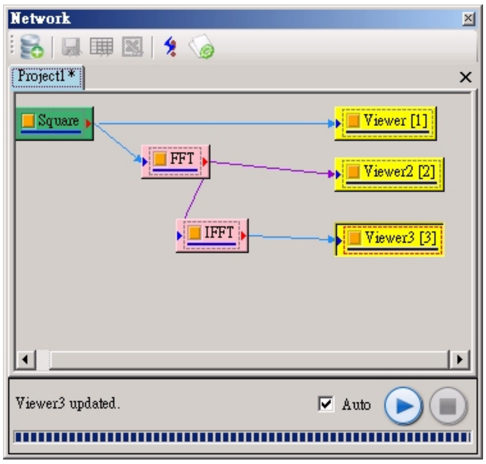
For the periodic boundary, one could decompose a signal $f(t)$ into a set of linear combination of sine and cosine waves ：

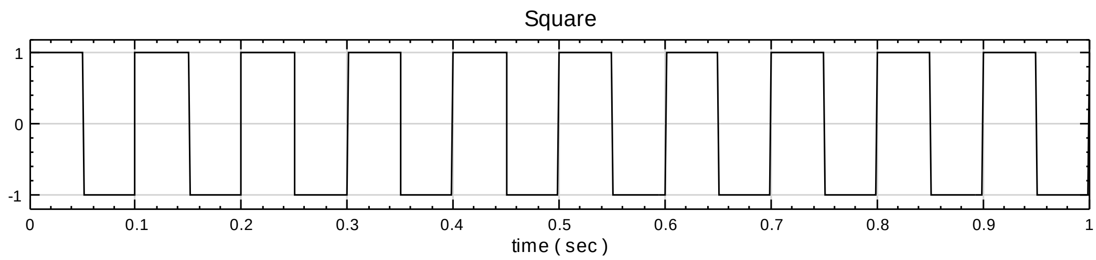
$$f(t)=\sum_{n=0}^{\infty} A_n \cos(n\frac{2\pi}{T}t)+B_n \sin(n\frac{2\pi}{T}t)$$ it is the Fourier series.

If the boundary be infinity, the $\sum$ is evolution to $\int$ and combining the sine and cosine with Eular formula($e^{ix}=\cos x+i\cdot\sin x$), that is Fourier transformation. The Fourier transformation is the right tool for non-periodic functions:

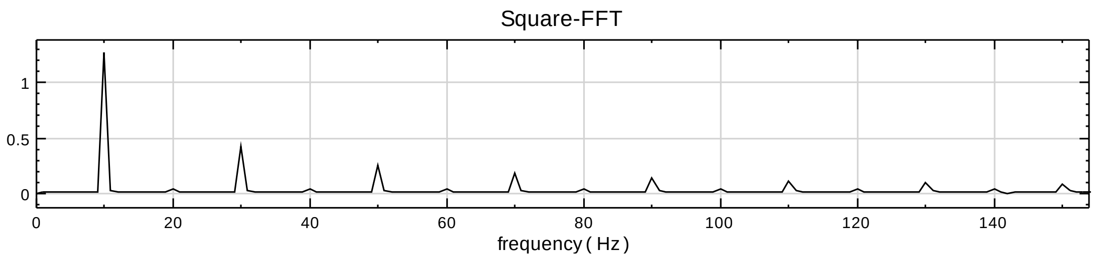
$$F(\omega)=F\left[f(t)\right]=\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$, the inverse be $$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$$

For example, the Square wave:

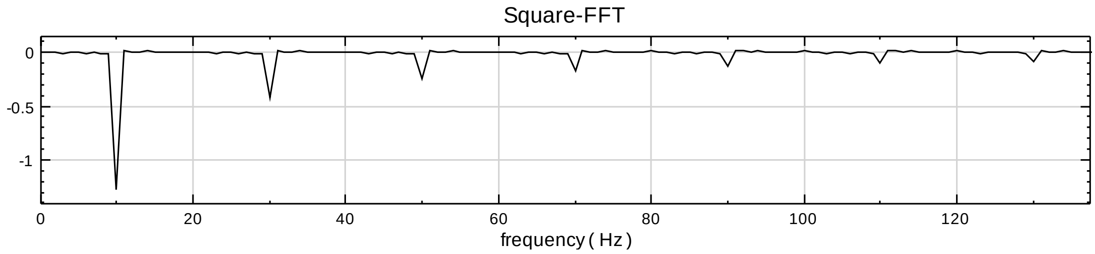




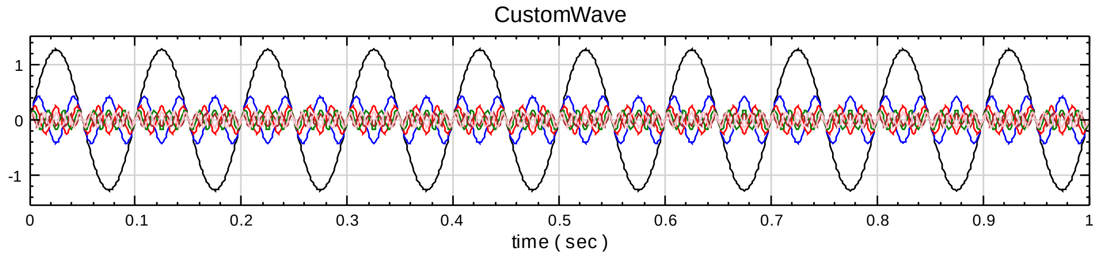
After Fourier transformation, the magnitude being:

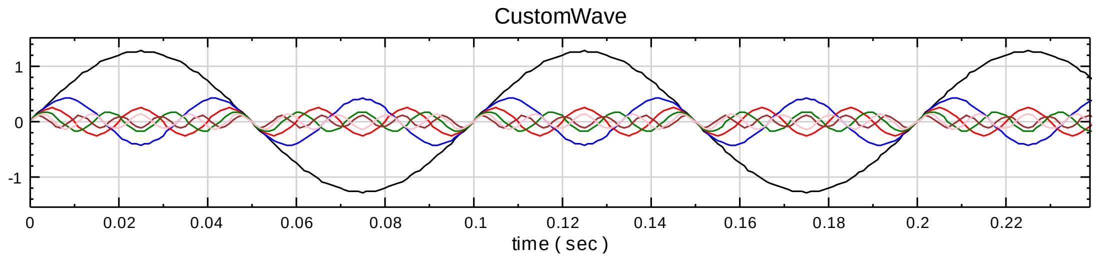


The imag.part be :

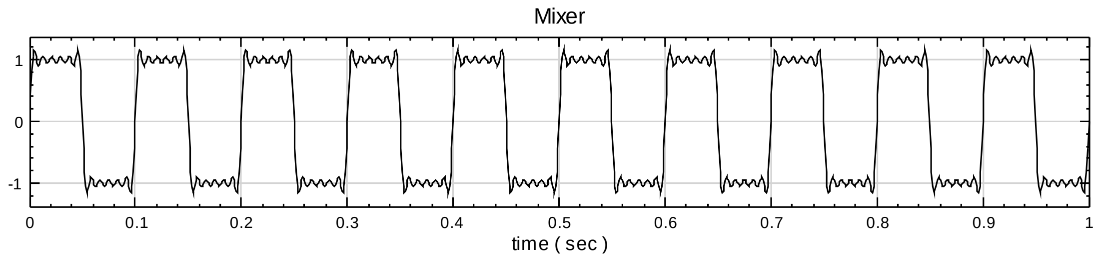


So the square wave could be decomposed into those sine /cosine wave by the Fourier result :



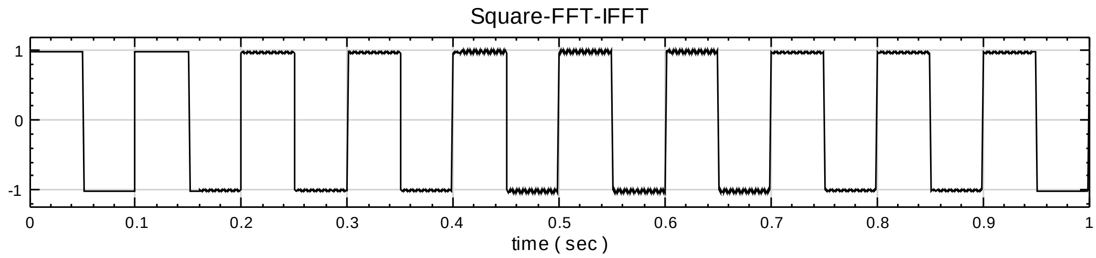


If we combine those wave by first 6 terms, the plot be :



It is Fourier inverse transformation.

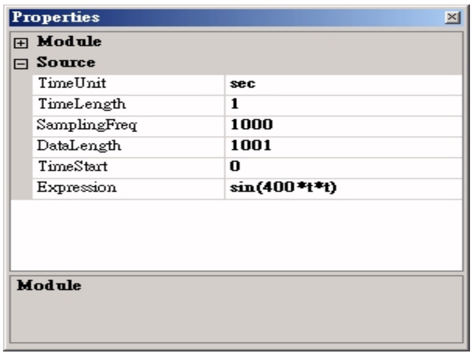
(The Inversing result takes by Visual Signal directly as there are about 30 terms-combination:

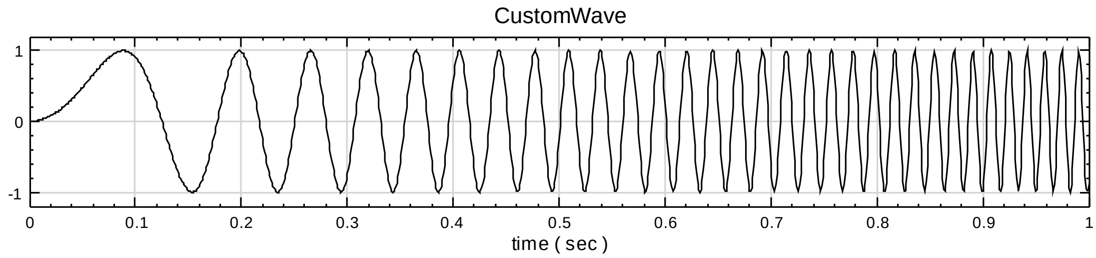


Take care the Gibbs phenomenia)

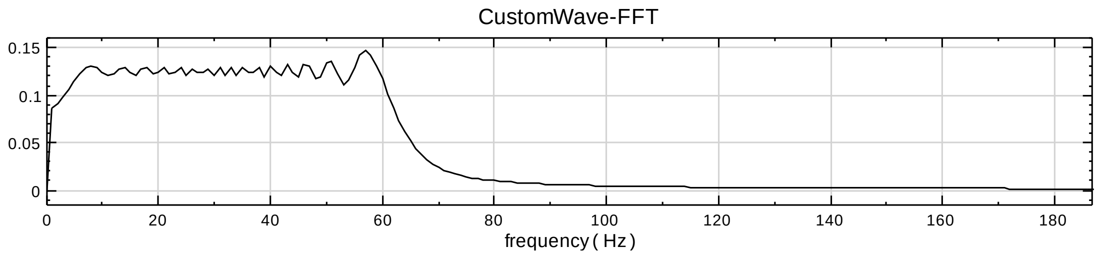
But the Fourier transformation is just good for **stationary** signal, but bad for general signal, i,e: $f(\omega(t))$ .The simplest case is a signal being $f(t)=\sin(\omega(t)\cdot t)$, $\omega(t)=a\cdot t$:

Custom wave set as:





its Fourier transformation :



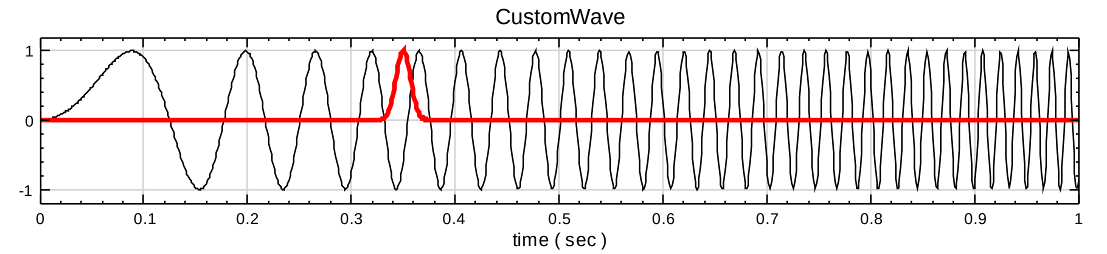
That does not represent the ω be proportion to t intuitively.

Short-Term Fourier

If we multiply the signal by a retard Gaussian $g(t-\tau)=\left(\frac{1}{\pi\sigma}\right)^{\frac{1}{2}} e^{-\frac{(t-\tau)^2}{\sigma}}$, (τ is the retard time , σ is the frequency resolution), then we get the short-term Fourier:

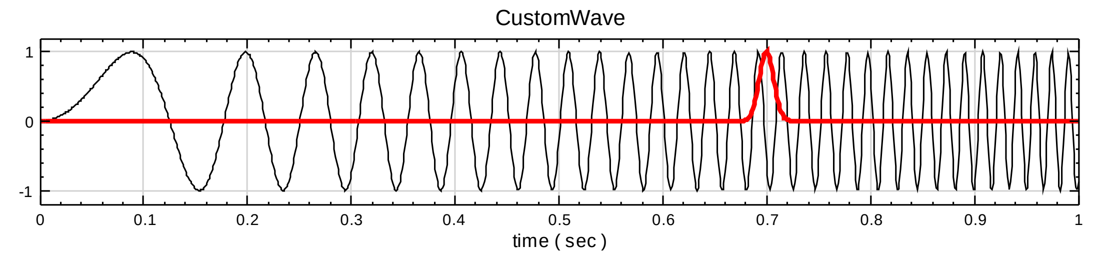
$$SF(\tau,\omega)=\int_{-\infty}^{\infty} f(t)\cdot\left(\frac{1}{\pi\sigma}\right)^{\frac{1}{2}} e^{\frac{-1}{\sigma}(t-\tau)^2}\cdot e^{i\omega t}dt$$

It means that, there is Gaussian “detector” in time τ to measure the signal. :

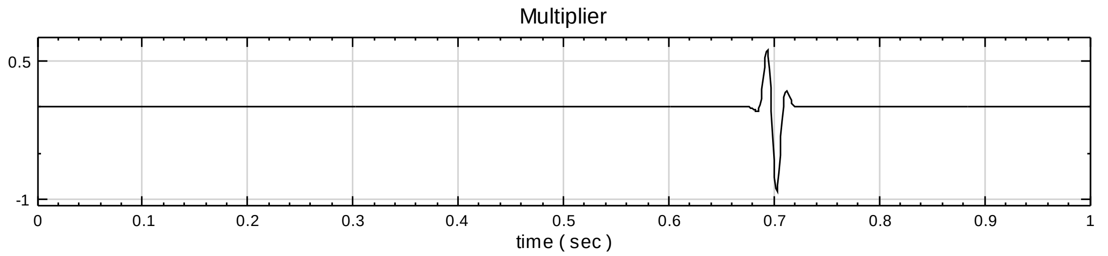


If we measure the signal by the Gaussian function at different τ , result is a function of measuring time and frequency(io angular velocity). it is what axis of the time-frequency plot are time and frequency.

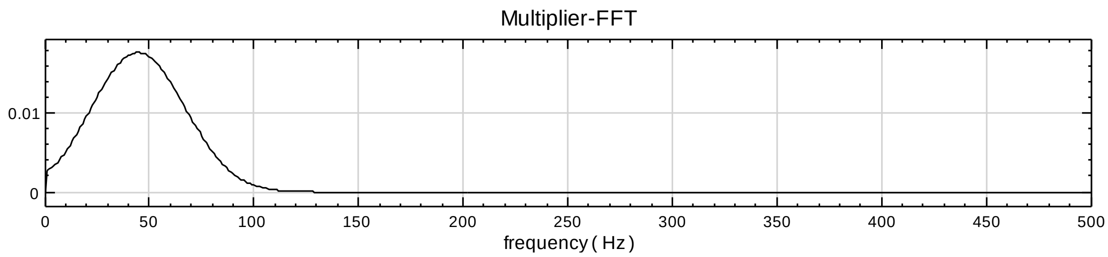
if the “detector ” in time $t^{'}$,the step to find $SF(\tau,\omega)\big|_{t^{'}=\tau}$,The step is :



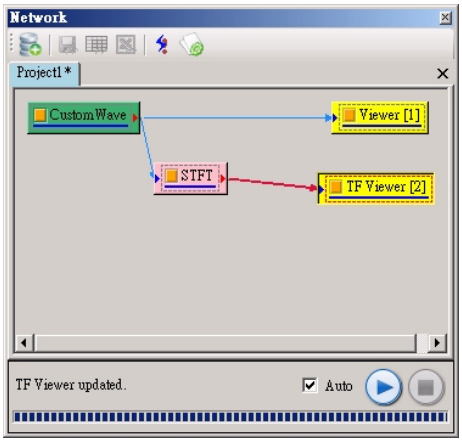
Multipling the detector by original signal :

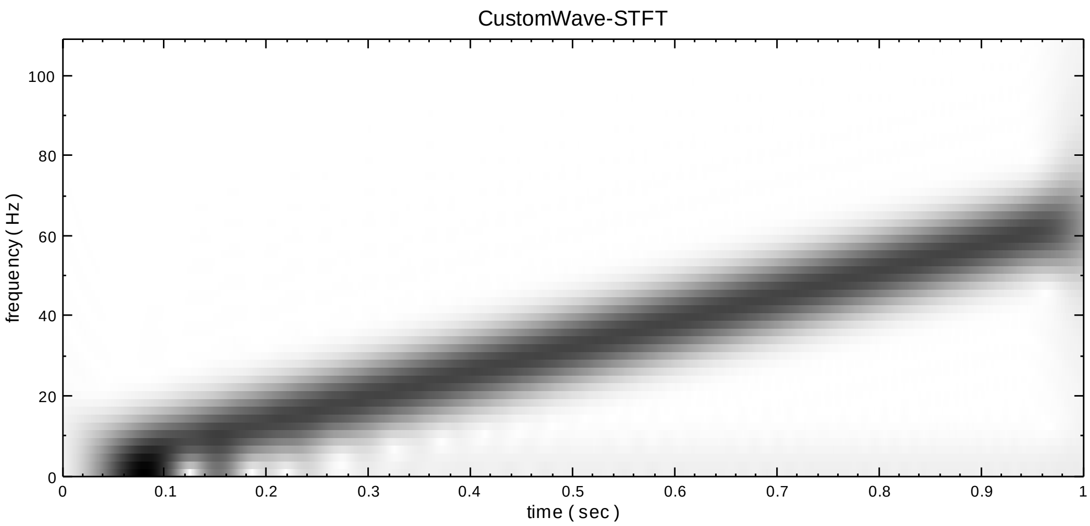


Fouriering it:



After above step for all time points, we could plot the $SF(\omega,\tau)$:

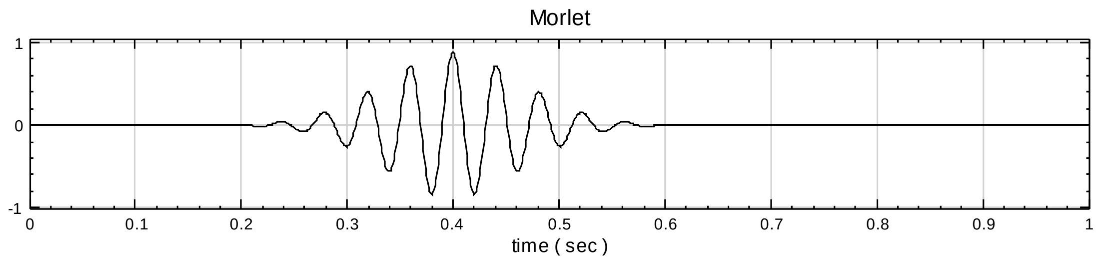




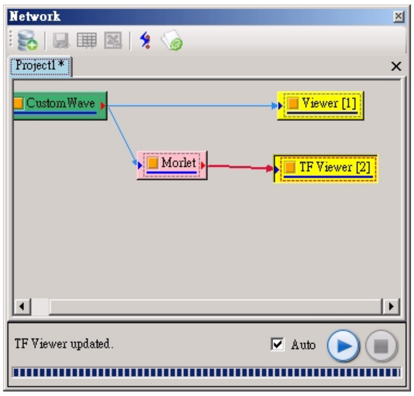
Wavelet transformation

Similar spirit, replace the gaussian multiply Fourier kernel($e^{i\omega t}$) term to wavelet kernel function $\phi_{a,\tau}(t) = \frac{1}{(a)^{\frac{1}{2}}} \cdot \phi(\frac{t-\tau}{a})$, ($\int_R \phi(t)dt = 1$ and a is the scale factor, $\frac{1}{a} \equiv frequency$), that is wavelet transformation:

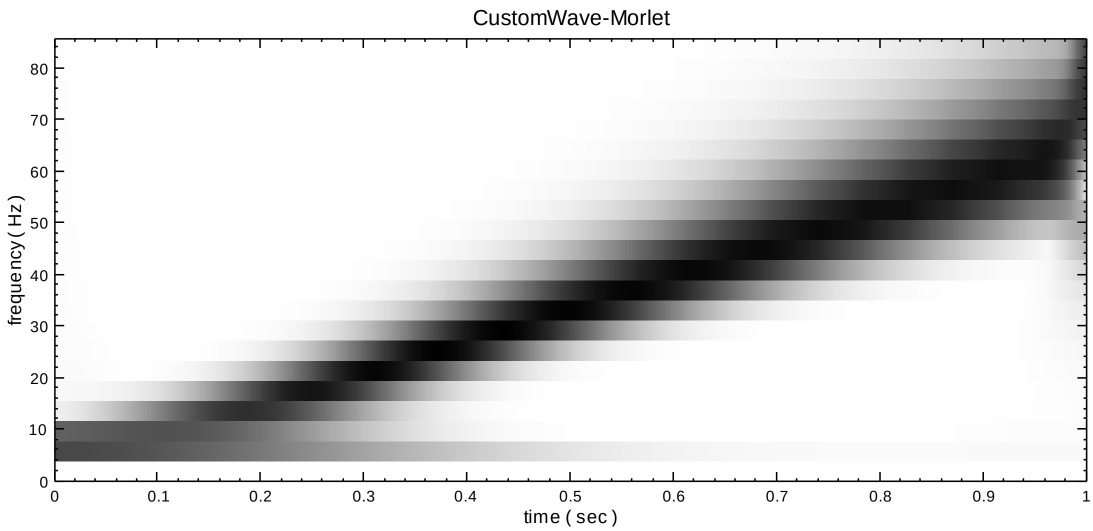
$WT(a,\tau) = \int_{-\infty}^{\infty} \frac{1}{(a)^{\frac{1}{2}}} \cdot \phi^*(\frac{t-\tau}{a}) \cdot f(t)dt$.(for example, the Morlet kernel is



)

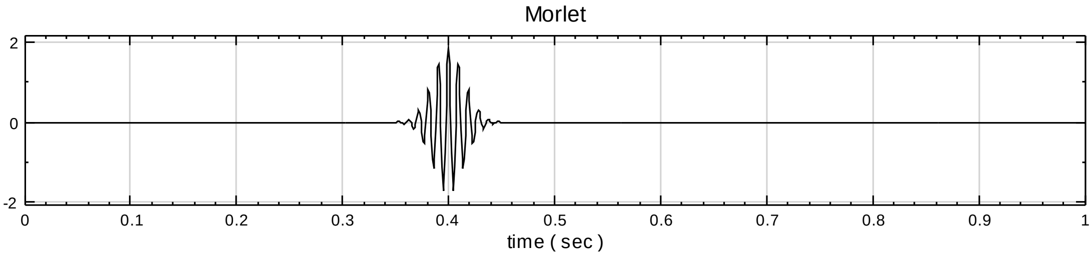


Wavelet transformation being:

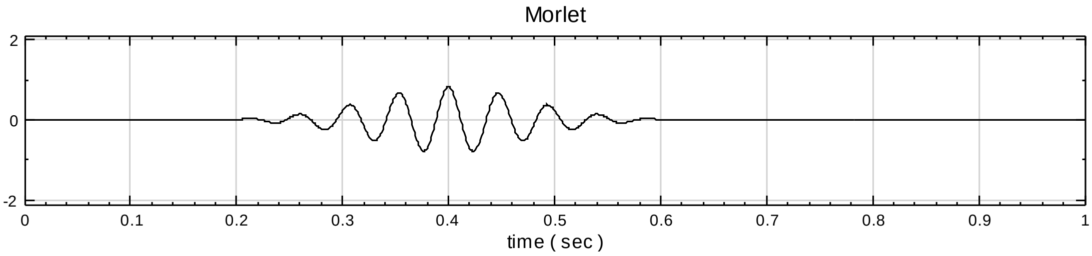


Noticing that the wavelet kernel function shape changes with frequency :

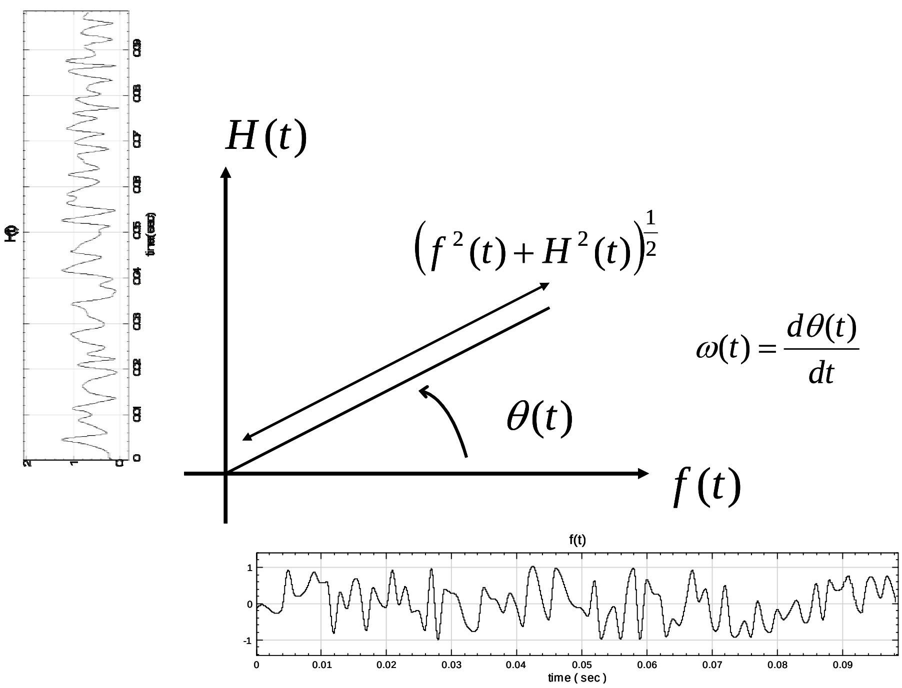
high frequency:



Low frequency



Hilbert transformation



Let original signal be $f(t)$, its Hilbert transformation is $H(t)$,called Hibert pair.

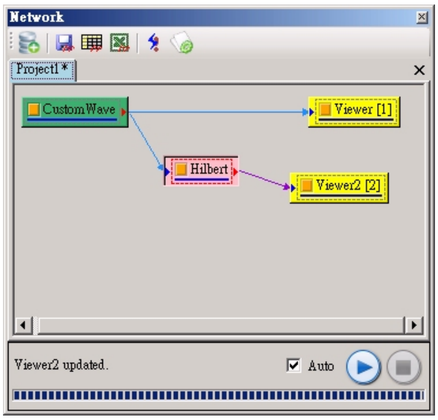
In mathematical expression, $H(t) = P.V.\,\frac{1}{\pi}\int_{-\infty}^{\infty} f(t')\left(\frac{1}{t'-\tau}\right)dt'\Big|_{t=\tau}$ (or written as the covolution form $H(t) = f(t) * \left(\frac{1}{t \cdot \pi} \right))$

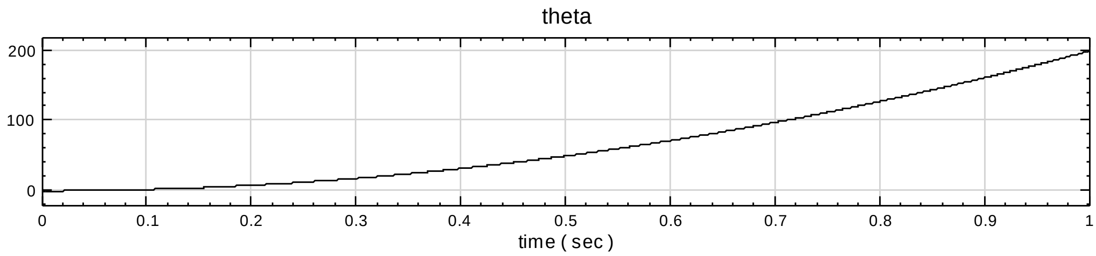
Let $Z(t) = f(t) + iH(t)$,also $Z(t)$ could represent as polar coordinates:

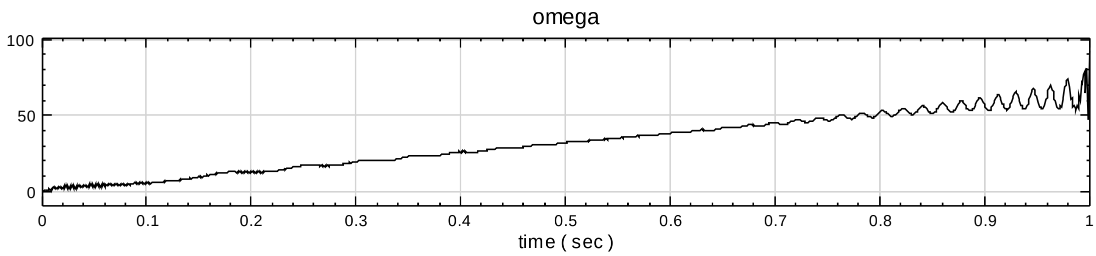
$Z(t) = a(t) \cdot e^{i\theta(t)}$, $a(t) = \left(f(t)^2 + H(t)^2 \right)^{\frac{1}{2}}$, $\theta(t) = \tan^{-1}\left(\frac{H(t)}{f(t)}\right)$

The frequency $\omega(t) \equiv \frac{d\theta(t)}{dt}$

We could calculate the phase $\theta(t)$ first to evolve the $\omega(t)$.







writing $Z(t) \to Z(t,\omega)$, $\mathrm{Re}(Z(t,\omega)) = f(t,\omega)$, $\mathrm{Im}(Z(t,\omega)) = H(t,\omega)$ .And finding the

$a(t,\omega) = \left(f(t,\omega)^2 + H(t,\omega)^2\right)^{\frac{1}{2}}$, that is Hilbert Spectrum.